DOI: 10.5604/01.3001.0014.0826

PRZEGLĄD STATYSTYCZNY
R. LXIV – ZESZYT 3 – 2017

WITOLD RZYMOWSKI[1], AGNIESZKA SUROWIEC[2]

# MODELLING POPULATION GROWTH WITH DIFFERENCE EQUATION METHOD

## 1. INTRODUCTION

The problem to create the demographic model to predict the world or countries population in a fixed time range has been investigated extensively for many years. The two oldest models: an exponential model by the British economist Thomas Malthus and a logistic one by the Belgian mathematician, Pierre Verhulst are the most known models in case of the world human population. Malthus framed a model based on the observation that biological populations, including human ones, tended to increase at rate proportional to the population size (Malthus, 1798). The model of Verhulst (Verhulst, 1838) differs from the Malthusian model by changing some assumption. This model was rediscovered and popularized in the 1920's by Pearl, Reed (1924). Both of these models, the Malthusian and the Verhulst model, really do not work in the longer periods of time (Murray, 1989). Malthus reasoned that an exponential growth of the world's population could not go on infinitely and therefore one must interrupt the inexorable working of model by artificially reducing the size of the population. The Verhulst model does not adequately describe either short-range changes or very long-range trends in human population growth. Pearl, Reed (1924) predicted a maximum world population of about 2 billion, which was exceeded by 1930.

In the work Smith (1977), in addition to above mentioned models, author examines the Doomsday model (Foerster et al., 1960). The unreliability of the Doomsday model consisting in the fact that the actual world population is slightly ahead of the Doomsday projection, nearly a generation after it was made, was observed in Austin, Brewer (1971) and again in Serrin (1975).

In the paper Rzymowski, Surowiec (2012) the authors propose a pseudologistic model of world population with three parameters estimated by the method which minimizes relative error. This model gives a better description of the world human population than the ones mentioned above.

[1] Lublin University of Technology, Fundamentals of Technology Faculty, Department of Applied Mathematics, 38 Nadbystrzycka St., 20-618 Lublin, Poland.

[2] Lublin University of Technology, Management Faculty, Department of Quantitative Methods in Management, 38 Nadbystrzycka St., 20-618 Lublin, Poland, corresponding author – e-mail: a.surowiec@pollub.pl.

All of these models of living beings are time dependent models and they tend to have the form:

$$L_t = f(t) + \varepsilon_t,\ t = 1,2,\ldots N. \tag{1}$$

$L_t$ is the function of time $t$ and it is the size of the population under consideration. $L_t$ can be considered as a dynamic system.

Therefore in this article we propose a new approach – the difference equation method – to obtain the time dependent models of the human population in countries or groups of countries[3]. We also consider the data representing the population in the world. We take into account the data from the years 1950–2011 ($N$ = 62), the data from the years 1950–2012 ($N$ = 63) and the data from the years 1950–2013 ($N$ = 64) depending on the country. The list of the countries and number of data corresponding them is presented in table 1.

Table 1.

The list of countries or group of countries and corresponding them the number of data $N$

| Country | $N$ | Country | $N$ |
|---|---|---|---|
| Australia | 64 | New Zealand | 64 |
| Austria | 64 | Norway | 64 |
| Belgium | 63 | Poland | 63 |
| Canada | 63 | Portugal | 64 |
| Chile | 64 | Slovak Republic | 64 |
| Czech Republic | 64 | Slovenia | 64 |
| Denmark | 64 | Spain | 64 |
| Estonia | 64 | Sweden | 64 |
| Finland | 64 | Switzerland | 62 |
| France | 64 | Turkey | 64 |
| Germany | 64 | United Kingdom | 64 |
| Greece | 63 | United States | 64 |
| Hungary | 64 | G7 | 64 |
| Iceland | 62 | OECD – Total | 64 |
| Ireland | 64 | World | 64 |
| Israel | 64 | Brazil | 63 |
| Italy | 64 | China | 64 |
| Japan | 64 | India | 64 |
| Luxembourg | 63 | Indonesia | 64 |
| Mexico | 64 | Russia | 64 |
| Netherlands | 64 | South Africa | 63 |

Source: own elaboration.

[3] Available at http:/stats.oecd.org (04 Mar 2016).

The difference equations have not been studied well to describe dynamic systems but many dynamic systems in many fields of science including physics, engineering, economics and biomedical sciences can be described by means of the second order differential equations (Li et al., 2002; Huang et al., 2006; Ramsay et al., 2007; Chen, Wu, 2008; Miao et al., 2009; Liang, Wu, 2008; Huang, 2010). The studies of differential equations in literature have mainly focused on the so-called forward problem, i.e., simulation and analysis of the behavior of state variables for a given system. The inverse problem, using the measurements of state variables to construct the econometric model and estimate the parameters that characterize the system, has not been studied well. Within the scope of investigation in this field an application example from human population dynamic study is tested in this article. The model of the number of living beings in chosen country in year $t$, $t = 1,2,\dots N$ (see eq. (1)) we consider as a discrete time model.

The aim of this work is to find the best model of living beings in chosen country. To estimate the parameters in our model of human population dynamic we use the least squares principle that are used comparatively often by mathematicians (Hemker, 1972; Bard, 1974; Li et al., 2005), computer scientists (Varah, 1982), and chemical engineers (Ogunnaike, Ray, 1994; Poyton et al., 2006). We also use the generalized Least Squares Method (Nowak, 2006; Rao, 1982).

The measure applied to verify the model of human population is the relative error. However, it should be pointed out that the empirical verification is only the correct final evaluation of the quality of the model.

## 2. THE DIFFERENCE EQUATIONS METHOD

Use of the second order differential equation is very popular to describe the dynamic systems in many fields of science. For example, the model:

$$x'' + 2\zeta\omega_0 x' + \omega_0^2 x = F_{ext}, \tag{2}$$

is the model of a damped harmonic oscillator, where $x$ is mass's position, $\frac{dx}{dt}$ is mass's velocity, $\frac{d^2x}{dt^2}$ is mass's acceleration, $\omega_0 = \sqrt{\frac{k}{m}}$ is called the undamped angular frequency of the oscillator and $\zeta = \frac{c}{2\sqrt{mk}}$ is called the damping ratio. In the model of a damped harmonic oscillator the coefficients $\omega_0$ and $\xi$ are positive. The equation (2) describes the behavior of the system where friction (frictional force $F_f = -cv$) or damping (damping force $F_d = -kx$) slows the motion of the system with known parameters $m$, $k$, $c$, where $m$ is mass, $k$ is spring constant and $c$ is called the viscous damping coefficient.

Let $L_t$ represent the number of living beings in chosen country in year $t$, $t = 1,2,\dots N$. Then the derivatives $L_t^{'}$, $L_t^{''}$ do not exist for our model (1). Therefore, we replace the differential equation (2) by the difference equation (3).

In analogy to the equation (2) we can consider the following equation for $L_t$:

$$L''_t = aL'_t + bL_t + c\,,\ t = 5,6,\dots,\tau\,, \tag{3}$$

where $a$, $b$, $c$ are the unknown parameters. $\tau = 10,11,\dots, N-4$.

Assuming (Lanczos, 1964):

$$L'_t = -\frac{1}{5}L_{t-2} - \frac{1}{10}L_{t-1} + \frac{1}{10}L_{t+1} + \frac{1}{5}L_{t+2}\,,\ t = 3,4,\dots, N-2\,, \tag{4}$$

$$L''_t = -\frac{1}{5}L'_{t-2} - \frac{1}{10}L'_{t-1} + \frac{1}{10}L'_{t+1} + \frac{1}{5}L'_{t+2}\,,\quad t = 5,6,\dots, N-4\,, \tag{5}$$

we can obtain the following difference model for $L_t$:

$$L_t = \alpha L_{t-1} + \beta L_{t-2} + \gamma L_{t-3} + \delta L_{t-4} + \zeta + \varepsilon_t\,,\ t = 5,6,\dots,\tau\,, \tag{6}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, $\zeta$ are the unknown parameters and $\varepsilon_t$ are the residuals. The model (6) is the discrete time model of $L_t$.

We estimate the parameters in model (6) with use the generalized Least Squares Method (Nowak, 2006; Rao, 1982).

To solve the equation (6) we consider the characteristic equation of this equation (6) in the form:

$$\lambda^4 - \alpha\lambda^3 - \beta\lambda^2 - \gamma\lambda - \delta = 0\,. \tag{7}$$

Depending on the values of parameters $\alpha$, $\beta$, $\gamma$, $\delta$ the equation (6) can have one of the ten forms of model of $L_t$ (Koźniewska, 1972; Sierpiński, 1946) but only three forms of model of $L_t$:

1. M1: $\hat{L}_t = C_1\lambda_1^t + C_2\lambda_2^t + C_3\lambda_3^t + C_4\lambda_4^t + C_5,$ (8)

   where $\lambda_1,\ \lambda_2, \lambda_3, \lambda_4 \in R,$

2. M2: $\hat{L}_t = C_1\lambda_1^t + C_2\lambda_2^t + C_3 r^t \cos(\varphi t) + C_4 r^t \sin(\varphi) + C_5,$ (9)

   where $\lambda_1,\ \lambda_2 \in R$ and $\lambda_3 = a + bi$, $\lambda_4 = a - bi$ and $b > 0$ are complex numbers and $r = \sqrt{a^2 + b^2}$, $\varphi = \arccos\frac{a}{r}$,

3. M3: $\hat{L}_t = C_1 r_1^t \cos(\varphi_1 t) + C_2 r_1^t \sin(\varphi_1 t) + C_3 r_2^t \cos(\varphi_2 t) + C_4 r_2^t \sin(\varphi_2 t) + C_5,$ (10)

   where $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are complex numbers: $\lambda_1 = a_1 + b_1 i$, $\lambda_2 = a_1 - b_1 i$, $\lambda_3 = a_2 + b_2 i$, $\lambda_4 = a_2 - b_2 i$, where $b_1$, $b_2 > 0$ and $r_1 = \sqrt{a_1^2 + b_1^2}$, $r_2 = \sqrt{a_2^2 + b_2^2}$, $\varphi_1 = \arccos\frac{a_1}{r_1}$, $\varphi_2 = \arccos\frac{a_2}{r_2}$

can be achieved numerically because it is very difficult to obtain zero or one numerically.

The parameters $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ can be estimated by using the Least Squares Method for chosen form of model of $L_t$ for $t = 5,6,\ldots,\tau$.

To verify the models obtained with the difference equations method we calculate for every $\tau = 10,11,\ldots, N-4$:

– the theoretical values of human population $\hat{L}_t(\tau)$, $t = 5,6,\ldots, N$ (a full range of data from the years 1950–2011 or 1950–2012, or 1950–2013) according to the country or group of countries (see table 1) with use M1, M2 or M3 model by equations (8), (9) or (10),
– the relative errors $\delta_t(\tau)$:

$$\delta_t(\tau) = \frac{\left|L_t - \hat{L}_t(\tau)\right|}{L_t} 100\%, \; t = 5,6,\ldots, N, \tag{11}$$

– the maximum relative errors $\delta_N(\tau)$ and $\delta_p(\tau)$ for the total and prediction range respectively:

$$\delta_N(\tau) = \max_{t=5,6,\ldots N} \delta_t(\tau), \tag{12}$$

$$\delta_p(\tau) = \max_{t=\tau+1,\tau+2,\ldots,N} \delta_t(\tau), \tag{13}$$

and

– $\tau^*$ that corresponds to

$$\delta_N(\tau^*) = \min_{\tau=10,11,\ldots,N-4} \delta_N(\tau), \tag{14}$$

where $\delta_N(\tau)$ is given by equation (12). The measure to verify the obtained model is $\delta_p(\tau^*)$. We find also $\tau^{**}$ such that $\delta_p(\tau) \le 5\%$ for $\tau \ge \tau^{**}$.

We tested the models defined by equation (6) for all the analyzed countries from table 1.

3. THE RESULTS FOR POPULATION MODELS

**The relative errors**

Table 2 shows values of $\tau^*$, models corresponding to them (see eq. (8), (9) and (10)), the relative errors $\delta_N\left(\tau^*\right)$ (see eq. (12) and (14)) and $\delta_p\left(\tau^*\right)$ (see eq. (13) and (14)) and $\tau^{**}$ for the human population for all analyzed countries, group of countries and for the world.

Table 2.

Values of $\tau^*$, models corresponding to them, the relative errors $\delta_N\left(\tau^*\right)$ and $\delta_p\left(\tau^*\right)$ and $\tau^{**}$ for the human population in analysed country

| Country | $\tau^*$ | Model | $\delta_N\left(\tau^*\right)$ [%] | $\delta_p\left(\tau^*\right)$ [%] | $\tau^{**}$ |
|---|---|---|---|---|---|
| Australia | 60 | M2 | 2.92 | 2.92 | 58 |
| Austria | 59 | M2 | 3.20 | 3.20 | 55 |
| Belgium | 24 | M2 | 3.76 | 3.76 | 59 |
| Canada | 59 | M1 | 2.18 | 2.18 | 47 |
| Chile | 60 | M1 | 1.95 | 1.95 | 39 |
| Czech Republic | 60 | M2 | 1.56 | 1.28 | 24 |
| Denmark | 59 | M2 | 2.61 | 2.61 | 53 |
| Estonia | 60 | M3 | 8.68 | 8.68 | - |
| Finland | 60 | M2 | 1.83 | 1.83 | 43 |
| France | 60 | M2 | 2.08 | 2.08 | 38 |
| Germany | 50 | M2 | 2.53 | 2.45 | 29 |
| Greece | 40 | M2 | 2.24 | 2.16 | 52 |
| Hungary | 60 | M3 | 4.11 | 4.11 | 54 |
| Iceland | 15 | M2 | 4.70 | 4.70 | - |
| Ireland | 60 | M2 | 4.87 | 3.52 | 59 |
| Israel | 57 | M2 | 7.45 | 1.96 | 48 |
| Italy | 60 | M2 | 4.41 | 4.41 | 59 |
| Japan | 58 | M3 | 2.65 | 2.40 | 50 |
| Luxembourg | 59 | M2 | 3.08 | 3.08 | 59 |
| Mexico | 56 | M1 | 2.53 | 2.16 | 47 |
| Netherlands | 54 | M2 | 0.87 | 0.78 | 38 |
| New Zealand | 57 | M2 | 4.13 | 4.06 | 54 |
| Norway | 59 | M2 | 0.76 | 0.54 | 58 |

| Country | $\tau^*$ | Model | $\delta_N(\tau^*)$ [%] | $\delta_p(\tau^*)$ [%] | $\tau^{**}$ |
|---|---|---|---|---|---|
| Poland | 58 | M3 | 1.68 | 1.65 | 45 |
| Portugal | 60 | M2 | 6.92 | 0.91 | 47 |
| Slovak Republic | 60 | M2 | 2.07 | 1.57 | 42 |
| Slovenia | 54 | M2 | 3.07 | 2.83 | 48 |
| Spain | 59 | M2 | 3.99 | 3.99 | 58 |
| Sweden | 60 | M2 | 2.45 | 2.45 | 52 |
| Switzerland | 58 | M2 | 5.93 | 5.93 | - |
| Turkey | 54 | M2 | 1.23 | 1.11 | 46 |
| United Kingdom | 60 | M1 | 4.66 | 4.66 | 59 |
| United States | 51 | M2 | 1.48 | 1.48 | 42 |
| G7 | 58 | M2 | 0.88 | 0.82 | 39 |
| OECD - Total | 59 | M2 | 0.37 | 0.37 | 30 |
| World | 53 | M3 | 0.55 | 0.55 | 44 |
| Brazil | 24 | M2 | 1.95 | 1.95 | 49 |
| China | 60 | M2 | 2.88 | 2.87 | 53 |
| India | 51 | M3 | 0.41 | 0.32 | 44 |
| Indonesia | 60 | M3 | 0.92 | 0.89 | 47 |
| Russia | 58 | M2 | 3.07 | 3.07 | 50 |
| South Africa | 56 | M3 | 1.21 | 1.21 | 54 |

Source: own elaboration.

The shaded cell in table 2 corresponds to the minimal value of $\delta_N\left(\tau^*\right)$ or $\delta_p\left(\tau^*\right)$ for all analyzed countries, group of countries and for the world. The minimal values of $\delta_N\left(\tau^*\right)$ and $\delta_p\left(\tau^*\right)$ are equal to 0.37 for OECD – Total and 0.32 for India, respectively.

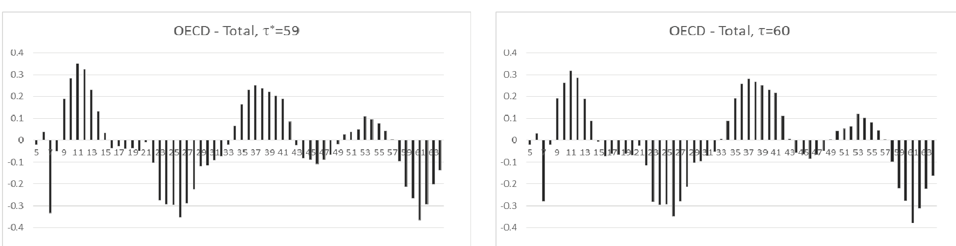


Figure 1. Comparison the $\delta_t$, $t = 5,6,\ldots, N$ for OECD-Total population for $\tau^*$ (left) and for $\tau = N - 4$ (right)

Source: own elaboration.

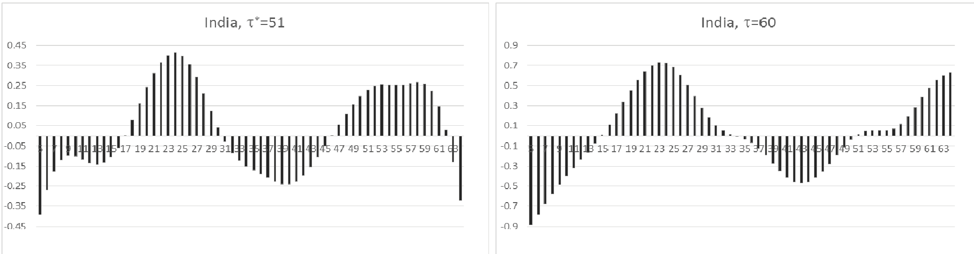


Figure 2. Comparison the $\delta_t$, $t$ = 5,6,…, $N$ for India population for $\tau^*$ (left) and for $\tau = N - 4$ (right)
Source: own elaboration.

Figure 1 and figure 2 presents the comparison between $\delta_t\left(\tau^*\right)$ and $\delta_t$ ($\tau = N - 4$) for OECD – Total and for India, respectively. The results presented in figure 1 for $\tau^* = 59$ and for $\tau = N - 4$ for OECD – Total are very similar. The models obtained for this group of countries are not very sensitive to change the $\tau$. OECD – Total is third in order after Czech Republic and Germany with small number of $\tau^{**}$. The figure 3 presents the values of $\delta_N\left(\tau\right)$ for $\tau$ = 30,31,…, $N - 4$ for OECD – Total. The best model for OECD – Total population is model of M2 type (see eq. (9)).

Comparing the results presented in figure 2 for India population we can find that the model obtained for $\tau^* = 51$ is better than the model obtained for $\tau = N - 4$ in spite of the longer period of estimation. The best model for India population is model of M3 type (see eq. (10)).

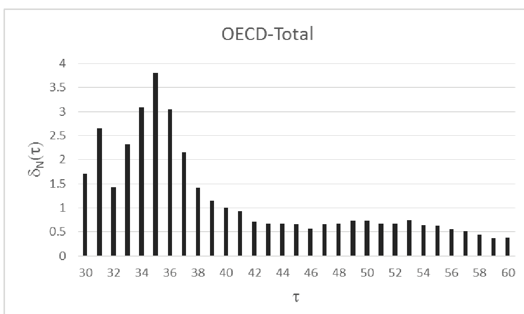


Figure 3. $\delta_N\left(\tau\right)$ for $\tau$ = 30,31,…, $N - 4$ for OECD – Total
Source: own elaboration.

Taking into account the value of $\delta_p\left(\tau^*\right)$ we can obtain 8 models out of 42 with $\delta_{\tau^*} < 1\%$. Only 2 models out of 42 have the value of $\delta_p\left(\tau^*\right)$ greater than 5%. In 30 countries out of 42 $\tau^{**} \leq 54$ what guarantees 10 years at least forecast with $\delta_p\left(\tau\right) \leq 5\%$ (see table 2).

Only in 3 country out of 42 (Estonia, Iceland and Switzerland, see table 2) $\tau^{**}$ do not exist.

**Remarks about the world population model**

Applying the difference equations method for the world population model we can obtain the model with very small maximal relative error. The minimal value of the maximal relative error for the world population model is less than 0.6% for $\tau^* = 53$ (see table 2) and for $\tau = 52$ and $\tau = 54$. It guarantees at least 10 years forecast with $\delta_p(\tau) \le 0.6\%$. In figure 4 we present the comparison between relative errors distributions for $\tau^* = 53$ and for $\tau = 54$. The models for world population for these values of $\tau$ are the models of M3 type (see eq. (10)). The models obtained for the world population are not very sensitive to change the $\tau$. It is the same result like this presented in figure 1 for OECD-Total. M3 type of model is obtained for world population for $\tau = 44,45,\ldots,57$. But if we consider the model for world population for $\tau = N - 4$ we will obtain the model of M2 type (see eq. (9)). In figure 5 we present the relative errors distributions for $\tau = N - 4$. It differs from this presented in figure 4.

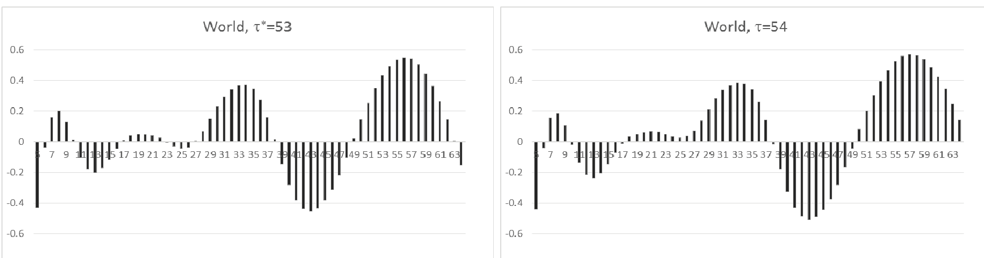


Figure 4. Comparison the $\delta_t$, $t = 5,6,\ldots, N$ for world population for $\tau^* = 53$ (left) and for $\tau = 54$ (right)

Source: own elaboration.

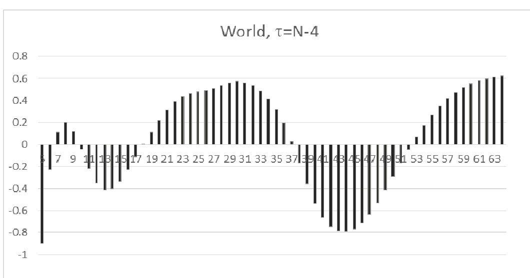


Figure 5. The distribution $\delta_t$, $t = 5,6,\ldots, N$ for world population for $\tau = N - 4$

Source: own elaboration.

The parameters of the models for $\tau^* = 53$, $\tau = 54$ and for $\tau = N - 4 = 60$ are presented in table 3.

Table 3.

The parameters of M2 and M3 models obtained for world population for $\tau^* = 53$, $\tau = 54$ and for $\tau = N - 4 = 60$

| Parameters | $\tau^* = 53$, (M3) | $\tau = 54$, (M3) | $\tau = N - 4 = 60$ (M2) |
|---|---|---|---|
| $\lambda_1$ | - | - | 0.993334625 |
| $\lambda_2$ | - | - | 0.978372676 |
| $r$ | - | - | 0.898565728 |
| $\varphi$ | - | - | 0.33155998 |
| $r_1$ | 1.007269262 | 1.002126621 | - |
| $\varphi_1$ | 0.025390024 | 0.023948069 | - |
| $r_2$ | 0.900067272 | 0.896320454 | - |
| $\varphi_2$ | 0.376079832 | 0.36475173 | - |
| $C_1$ | -1233332453 | -2446916724 | -26107791799 |
| $C_2$ | 2127947086 | 2099598613 | 5630888596 |
| $C_3$ | -59462245.29 | -65658485.65 | -99683889.48 |
| $C_4$ | 22589702.33 | 32577674.82 | 87080983.82 |
| $C_5$ | 3667243839 | 4875343817 | 22834463029 |

Source: own elaboration.

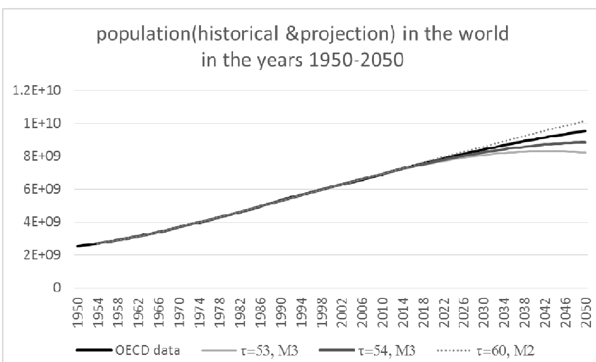


Figure 6. Comparison between the M2, M3 models and the world data for projection available at http:/stats.oecd.org, 2016

Source: own elaboration.

The comparison between the models obtained for $\tau^* = 53$, $\tau = 54$ and for $\tau = N - 4 = 60$ with parameters presented in table 3 and the world data for projection[4]

[4] Available at http://stats.oecd.org, 2016.

is shown graphically in figure 5. The forecast obtained by M3 models satisfies better the statement "There must be an upper limit on the earth's life support capabilities, and therefore the population cannot grow without bound" (Robertson et al., 1961) than OECD model. Our model satisfies another statement "A model predicts that the world's population will stop growing in 2050"[5].

## 4. FINAL REMARKS

a) The difference equations method provides an easy programmable way to choose the model of the population in particular countries in the world as well as for the world population. The type of the model (see eq. (8), (9) and (10)) is established and depending on the values of parameters of characteristic equation (7) of equation (6).
b) Despite the statement that population forecasting by fitting mathematical curves is notably unreliable because it ignores so many important factors of demography (Dorn, 1962), the model of population obtaining by the difference equations method provides remarkably good fit with nearly all available data. For nearly all analyzed countries (40 out of 42), the $\delta_p\left(\tau^*\right)$ is less than 5%, for the world $\delta_p\left(\tau^*\right) < 0.6\%$.
c) The M1 model for $\left(\tau^*\right)$ (see eq. (8) and table 2) was obtained only for 4 countries. It is not a surprise that the population cannot be modelled by the model of *t* that estimates the number of people without upper limit (Robertson et al., 1961)[6].

## REFERENCES

Austin A. L., Brewer J. W., (1971–1972), World Population Growth and Related Technical Problems, *Technological Forecasting and Social Change,* 3, 23–49.

Bard Y., (1974), *Nonlinear Parameter Estimation*, Academic Press, New York.

Chen J., Wu H., (2008), Efficient Local Estimation for Time-varying Coefficients in Deterministic Dynamic Models with Applications to HIV-1 Dynamics, *Journal of the American Statistical Association*, 103, 369–384.

Dorn H. F., (1962), World Population Growth: An International Dilemma, *Science*, 135, 283–290.

Hemker P. W., (1972), Numerical Methods for Differential Equations in System Simulation and in Parameter Estimation, in: Hemker H. C., Hess B., (eds.), *Analysis and Simulation of Biochemical Systems*, North Holland Publ. Comp, 59–80.

Huang Y., (2010), A Bayesian Approach in Differential Equation Dynamic Models Incorporating Clinical Factors and Covariates. *Journal of Applied Statistics*, 37 (2), 181–199.

Huang Y., Liu D., Wu H., (2006), Hierarchical Bayesian Methods for Estimation of Parameters in a Longitudinal HIV Dynamic System, *Biometrics*, 62 (2), 413–23.

Koźniewska I., (1972), *Równania rekurencyjne,* PWN, Warszawa.

[5] http://www.sciencedaily.com/releases/2013/04/130404072923.htm, 2013.

[6] http://www.sciencedaily.com/releases/2013/04/130404072923.htm, 2013.

Lanczos C., (1964), *Applied Analysis,* Prentice Hall. Inc., USA.

Li L., Brown M. B., Lee K. H., Gupta S., (2002), Estimation and Inference for a Spline-Enhanced Population Pharmacokinetic Model, *Biometrics*, 58 (3), 601–611.

Li Z., Osborne M. R., Prvan T., (2005), Parameter Estimation of Ordinary Differential Equations, *IMA Journal of Numerical Analysis*, 25 (2), 264–285.

Liang H., Wu H., (2008), Parameter Estimation for Differential Equation Models Using a Framework of Measurement Error in Regression Models, *Journal of the American Statistical Association*, 103 (484), 1570–1583.

Malthus T. R., (1798), *An Essay on the Principal of Population*, J. Johnson, in St. Paul's Church-Yard: London.

Miao H., Dykes C., Demeter L. M., Wu H., (2009), Differential Equation Modeling of HIV Viral Fitness Experiments: Model Identification, Model Selection, and Multimodel Inference, *Biometrics*, 65 (1), 292–300.

Murray J. D., (1989), *Mathematical Biology*, 19 of Biomathematics Texts, Springer: Berlin.

Nowak E., (2006), *Zarys metod ekonometrii. Zbiór zadań,* PWN, Warszawa.

Ogunnaike B. A., Ray W. H., (1994), *Process Dynamics, Modeling, And Control*, Series: Topics in Chemical Engineering, Oxford University Press, New York.

Pearl R., Reed L. J., (1924), The Growth of Human Population, in: Pearl R., (ed.), *Studies in Human Biology*, Wiliam and Wilkins: Baltimore, 584–637.

Poyton A. A., Varziri M. S., McAuley K. B., McLellan P. J., Ramsay J. O., (2006), Parameter Estimation in Continuous-Time Dynamic Models Using Principal Differential Analysis, *Computer and Chemical Engineering*, 30 (4), 698–708.

Ramsay J. O., Hooker G., Campbell D., Cao J., (2007), Parameter Estimation for Differential Equations: A Generalized Smoothing Approach (with Discussions), *Journal of the Royal Statistical Society*: Series B, 69 (5), 741–796.

Rao C. R., (1982), *Modele liniowe statystyki matematycznej*, PWN, Warszawa.

Robertson J. S., Bond V. P., Cronkite E. P., Hutton W. E., Howland W. E., Shinbrot M., von Foerster H., Mora P. M., Amiot L. W., (1961), Doomsday, *Science*, 133, 936–946.

Rzymowski W., Surowiec A., (2012), Method of Parameters Estimation of Pseudologistic Model, in: Zieliński Z. E., (ed.), *Rola informatyki w naukach ekonomicznych i społecznych. Innowacje i implikacje interdyscyplinarne,* 2, WSH, Kielce, 256–265.

Serrin J., (1975), Is 'Doomsday' on target? (Letter), *Science*, 189, 86–88.

Sierpiński W. F., (1946), *Zasady algebry wyższej,* PWN, Warszawa–Wrocław.

Smith D. A., (1977), Human Population Growth: Stability or Explosion? *Mathematics Magazine*, 50 (4), 186–197.

Varah J. M., (1982), A Spline Least Squares Method for Numerical Parameter Estimation in Differential Equations, *SIAM Journal on Scientific and Statistical Computing*, 3 (1), 28–46.

Verhulst P. F., (1838), Notice sur la loi que la population suit dans son d'accroissement. *Correspondance Mathématique et Physique Publiée par A. Quételet*, 10, 113–121.

von Foerster H., Mora P. M., Amiot L. W., (1960), Doomsday: Friday, 13 November, A. D. 2026, *Science*, 132 (3436), 1291–1295.

MODELOWANIE LICZBY LUDNOŚCI ZA POMOCĄ RÓWNAŃ RÓŻNICOWYCH

Streszczenie

W tym artykule przedstawiono metodę równań różnicowych czwartego rzędu do doboru postaci modelu ekonometrycznego. Zastosowanie zaproponowanej metody pozwala uzyskać model dobrze dopasowany do danych empirycznych o małym maksymalnym błędzie względnym.

Metoda równań różnicowych czwartego rzędu w artykule została wykorzystana do modelowania liczby ludności świata jak również do modelowania liczby ludności w wybranych krajach świata.

**Słowa kluczowe:** równania różnicowe, modele nieliniowe, estymacja parametrów strukturalnych, maksymalny błąd względny modelu, demografia

MODELLING POPULATION GROWTH
WITH SECOND ORDER DIFFERENTIAL EQUATION METHOD

Abstract

In this paper, we present a new method of the econometric model construction: the difference equation method. We illustrate the proposed approach using an application example from human population dynamic study. We find out that proposed method is very useful to find one of the three forms of proposed models of human population satisfying the small maximal relative errors. The maximal relative error is a measure to verify the model of human population.

The proposed method is tested for all available data referring to the human population in the OECD countries as well as in selected non-OECD countries.

**Keywords:** difference equations, nonlinear models, parameter estimation, relative error, demography